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The Zero Discounting and Maximin Optimal Paths in a Simple Model of Global Warming

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Abstract

Following Stollery[1998], we extend the Solow, Dasgupta-Heal model to analyze the effects of global warming The rise of temperature is caused by the use of fossil resources so that the temperature level can be linked to the remaining stock of these resources. The rise of temperature affects both productivity and utility. We characterize optimal solutions for the maximin and zero-discounting cases and present closed form solutions for the case where the production function and utility function are Cobb-Douglas, and the temperature level is an exponential function of the remaining stock of resources. We show that a greater weight of temperature in the preferences or a larger intertemporal elasticity of substitution both lead to postpone resource use.

JEL Classification: Q32

1 Introduction

Global warming and its consequences for the welfare of future generations appear more and more as one of the foremost economic issues asking for proper policies. The depletion of the

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stock of fossil resources plays a major role in this phenomenon. A policy such as a tax on hydrocarbon-burning is necessary to control the depletion of the stock of resource and the ensuing rise of temperature. To a large extent, this amounts to a simple reallocation of carbon dioxide emissions from today to the future. Its consequences for the welfare profile of the various generations is quite important however. Capital accumulation and technical progress may mitigate the consequences for future generations of the depletion of natural resources. There is no doubt however that laisser-faire will not yield optimal results and that government intervention is required.

The design of the optimal tax response to climate change requires the choice of a social criterion to weight the welfare levels of current and future generations. The maximin criterion appears a natural candidate as this Rawlsian criterion ensures in practice that each generation enjoys the same level of utility. An alternative is the zero discounting criterion (Ramsey [1928]), which has often been considered in the natural resources and environmental economics literature, starting with Solow [1974]. This criterion rests on the ethical first principle that different generations should have equal weights in social welfare. This does not preclude, however, the admission of some substitution between the welfare levels of the different generations. The unavoidable reduction of the stock of oil a priori tends to disadvantage future generations. Even in the absence of technical progress, capital accumulation however exerts a compensating influence. As has been shown by Dasgupta and Heal [1979] or Mitra [1980], consumption and therefore welfare may eventually be forever rising. In this sense, the feasible domain is biased in favor of future generations, rather than to their detriment. In any case, even if society puts equal weights on the different generations' welfare, it has to choose to what extent it is ready to substitute current welfare for future welfare.

This zero discounting criterion is more general than the maximin criterion. It includes maximin as a particular case, and d'Autume and Schubert [2008a] stressed that this last criterion may be recovered as the limit case of a zero intertemporal elasticity of substitution.

Ken Stollery [1998] developed a global warming version of the Solow [1974] maximin model[12]. Current temperature was directly related to cumulative oil extraction, and therefore to the current remaining stock of resource. This simplifying assumption avoids the explicit modeling of the accumulation of carbon dioxide in the atmosphere[3]. Temperature, which rising harms both productivity and welfare, entered negatively in current goods production and the current utility of a representative agent. A Pigovian tax internalized these negative externalities associated with current oil extraction. Two interesting results emerged: (a) given the proper Pigovian tax in place, the investment of oil rents in new produced man-made capital sustained a constant utility program and (b) a closed-form solution got developed for the Cobb-Douglas case when temperature only affects the production function.

We generalize these results in two directions.

Building on d'Autume and Schubert [2008a], we present a more general solution method which also applies to the case in which temperature affects current utility negatively as well as current goods production. We then lay out the closed-form solution for the case where production and utility are Cobb-Douglas and the function linking temperature to the remaing stock of resource is exponential, as in Stollery [1998].

We are thus able to examine how a representative consumption path changes when the parameter linking temperature to utility increases. The larger this parameter, the more gradual the temperature rise is over the long run.

For the simpler case of temperature only affecting current goods production negatively, Stollery was able to show that current investment in produced capital was unchanging, as in Solow [1974]. We show that including the effect on utility leads current investment in produced capital to rise toward a bound.

In the second place we consider the more general case of a zero discounting social criterion. Following here d'Autume and Schubert [2008b], we show that a generalized Hartwick [1977] rule applies. In the maximin case, total investment, including the value of the depletion of the stock of resource, should be constant. In the zero discounting case, total investment should be increasing. As Ramsey [1928] first showed in a pure growth framework, the new rule is that the level of investment, evaluated in terms of utility, should depend on the distance between the current and the long run value of utility.

We again provide closed-form solutions for the specific case considered in the first part of the paper.

2 Stollery's model and Maximin

2.1 The *invest resource rents* rule

The objective function in Stollery's analysis is the maximization of $u(c(t), T(X(t))$ subject to u constant, for $c(t)$ aggregate consumption, $T(.)$ temperature, and $X(t)$ the remaining stock of oil. Hence derivative, $T'(X)$ is negative. Current extraction is $x(t)$ and $x(t) = -\dot{X}(t)$. Utility is assumed to be increasing in C and decreasing in T. Temperature rises with cumulative extraction. Hence as $X(t)$ declines, $T(X(t))$ increases. Population, in the background, is assumed to be unchanging. We leave labor (population) out of the production function. On the production side, we have $Y(t) = F(K(t), x(t), T(X(t)))$ leading to the account

$$\dot{K}(t) = F(K(t), x(t), T(X(t))) - c(t).$$

Temperature also has a negative effect on production in this formulation.

The Hotelling Rule for this model must incorporate the negative impact of current oil extraction on the temperature variable. This is dealt with by a first best or Pigovian tax on current extraction. More on this below. The Hotelling Rule (dynamic efficiency condition for extraction) for our problem[4] with negative extraction externalities is (dropping the time index)

$$\frac{\dot{q}}{q} = F_K - \left\{ F_T + \frac{u_T}{u_c} \right\} \frac{T'}{q} \tag{1}$$

for

$$q = F_x. \tag{2}$$

The extra term, $-\left\{ F_T + \frac{u_T}{u_c} \right\} \frac{T'}{q}$ is a negative rate, a "tax", which means that a unit extracted receives current price q, and when this is invested for a period, earns only $F_K - \left\{ F_T + \frac{u_T}{u_c} \right\} \frac{T'}{q}$. The "tax" term is negative. It is as if the government has imposed a low ceiling on the rate of return earnable by an extractor. This reduced rate of return to extraction implies slower extraction in simple, partial equilibrium models and of course this is the desired impact of a Pigovian tax in this model. $\left\{ F_T + \frac{u_T}{u_c} \right\} T'$ is also the marginal damage induced by oil extraction and burning. The shadow price of oil is the sum of future marginal damages discounted at the "market rate" (the marginal product of produced capital):

$$q(t) = \int_t^\infty e^{-\int_t^s F_K(\tau)d\tau} \left\{ F_T + \frac{u_T}{u_c} \right\} T'(X(s))ds.$$

Introduce now the "invest resource rents" rule:

$$\dot{K} = xq. \tag{3}$$

Together with the Hotelling Rule, the invest rents rule yields a constant utility path in this model. To see this, we note first that $d\dot{K}/dt = \dot{x}q + x\dot{q}$ and $\dot{q}$ in turn is defined in the Hotelling Rule in (1). That is

$$\frac{d\dot{K}}{dt} = \dot{x}q + x \left[qF_K - \left\{ F_T + \frac{u_T}{u_c} \right\} T' \right].$$

We can use this in the calculation of $\dot{u} = u_c\dot{c} + u_T\dot{T}$. $\dot{u}$ should be zero for a maximin program. Hence we turn first to evaluate $\dot{c}$ from the accounting relation $c = F(L, x, T) - \dot{K}$. That is, we consider

$$\dot{c} = F_K\dot{K} + q\dot{x} + F_T\dot{T} - \frac{d\dot{K}}{dt}. \tag{4}$$

We proceed to substitute for $\dot{K} = xq$, $\frac{d\dot{K}}{dt}$ from above and $\dot{T} = -T'x$. We obtain $\dot{c} = \frac{u_T}{u_c}T'x$. This implies that $\dot{u} = 0$.

Hence Hotelling Rule, adjusted with a Pigovian tax on extraction activity, and "invest resource rents" imply $\frac{du(.)}{dt} = 0$. The novelty here is that Pigovian taxes are needed to sustain the optimum and "invest resource rents" must be carried out with optimal prices, prices inclusive of Pigovian taxes.

The converse to the result above is also true and the demonstration is also quite direct. Given $\frac{du(.)}{dt} = 0$ and Hotelling's Rule inclusive of the Pigovian tax, we obtain $G = 0$, for $G = \dot{K} + \dot{X}q$, G standing for "Genuine Savings".

We start with $\dot{u} = 0$ or $\dot{c} = -\frac{u_T}{u_c}T'x$. We also have $\dot{G} = \frac{d\dot{K}}{dt} - x\dot{q} - q\dot{x}$, given the definition of G. We now consider $\dot{c}$ from the accounting relation, $c = F(K, x, T) - \dot{K}$ (see (4) above) and substitute our two relations, namely $\dot{c}$ and $\frac{d\dot{K}}{dt}$ (in $\dot{G}$); we obtain

$$\dot{G} - GF_K = F_K qx - \left\{ F_T + \frac{u_T}{u_c} \right\} T'x - \dot{q}x.$$

The right hand side is zero, given extraction satisfying our Hotelling Rule. Hence $\dot{u} = 0$ and our Hotelling Rule imply $\dot{G} - GF_K = 0$. This property, $\dot{G} - GF_K = 0$ has been shown by Dixit, Hammond and Hoël [1980] to imply $G = 0$ for maximin paths. Hence we infer that indeed $G = 0$ for our dynamically efficient, maximin economy, where dynamic efficiency in extraction incudes a Pigovian tax.

2.2 Closed Form Solutions

Stollery perceived that for the special case of the temperature externality only in the production function, the solution for the Cobb-Douglas case involved $K(t)$ linear in time. Temperature does not appear in utility, and he therefore had a constant consumption scenario. He assumed the following explicit functional forms:

$$Y(t) = K(t)^{\alpha} x(t)^{\beta} T(X(t))^{-\gamma} \tag{5}$$

and

$$T(X(t)) = \overline{T} e^{-\phi X(t)}, \tag{6}$$

$\alpha, \beta, \gamma, \phi > 0$, $\alpha + \beta < 1$, $\alpha > \beta$.

$\overline{T}$ is the maximal temperature reached when oil is totally exhausted. Stollery was then able to get a complete solution:

$$K(t) = K_0 + \frac{\beta c^*}{1-\beta} t$$

where the sustainable level of consumption is

$$c^* = (1-\beta) \left(\frac{\beta(\alpha-\beta)}{\phi\gamma} \right)^{\frac{\beta}{1-\beta}} \overline{T}^{-\frac{\gamma}{1-\beta}} (e^{\phi \frac{\gamma}{\beta} X_0} - 1)^{\frac{\beta}{1-\beta}} K_0^{\frac{\alpha-\beta}{1-\beta}},$$

and is associated to the following path of the resource stock:

$$X(t) = \frac{\beta}{\phi\gamma} \ln(1 + BK(t)^{\frac{-(\alpha-\beta)}{\beta}}) \quad \text{with } B = K_0^{\frac{\alpha-\beta}{\beta}} (e^{\phi\frac{\gamma}{\beta}X_0} - 1).$$

We turn to the case where current temperature affects current utility negatively and, to this end, follow d'Autume and Schubert [2008a] method.

The system characterizing the optimal maximin path[5] is composed of Hotelling Rule (1), equation (2), "invest resource rents" (3) and $u(c, T(X)) =$ constant.

For the Cobb-Douglas production function (5), "invest resource rents" becomes $\dot{K} = \beta Y$, and we have $c = (1-\beta)Y$. The differential system is:

$$\begin{aligned} \dot{K} &= \beta Y && (7) \\ \dot{X} &= -Y^{1/\beta} K^{-\alpha/\beta} T(X)^{\gamma/\beta} && (8) \\ Y &= \frac{C(u, T(X))}{1-\beta}, && (9) \end{aligned}$$

where $C(u, T(X))$ is the utility function, "inverted". We then substitute Y, eliminate time and separate variables to get

$$-\left(\frac{C(u, T(X))}{1-\beta}\right)^{-\frac{1-\beta}{\beta}} T(X)^{-\frac{\gamma}{\beta}} dX = \frac{1}{\beta} K^{-\alpha/\beta} dK.$$

This integrates to

$$\int_X^{X_0} \left(\frac{C(u, T(\xi))}{1-\beta}\right)^{-\frac{1-\beta}{\beta}} T(\xi)^{-\frac{\gamma}{\beta}} d\xi = \frac{1}{\beta} \int_{K_0}^{K} \kappa^{-\alpha/\beta} d\kappa. \qquad (10)$$

This relation rests on the assumption of a Cobb-Douglas production function, which yields the simple investment rule $\dot{K} = \beta Y$, but holds for any $u(c, X)$ utility and $T(X)$ temperature functions. It could be used to derive general properties of the optimal path, depending of the properties of the utility and temperature functions[6]. We rather focus on the simple case considered by Stollery, where the utility function is also Cobb-Douglas:

$$u(c, T(X)) = cT(X)^{-\varepsilon} \text{ or } C(u, T(X)) = uT(X)^{\varepsilon}, \quad \varepsilon \geqslant 0. \tag{11}$$

Then (10) becomes, for a constant u,

$$\left(\frac{u}{1-\beta}\right)^{-\frac{1-\beta}{\beta}} \int_X^{X_0} T(\xi)^{-\frac{\varepsilon(1-\beta)+\gamma}{\beta}} d\xi = \frac{1}{\alpha-\beta}\left(K_0^{-\frac{\alpha-\beta}{\beta}} - K^{-\frac{\alpha-\beta}{\beta}}\right). \tag{12}$$

Given the temperature function (6), which we write

$$T(X) = T_0 e^{\phi(X_0 - X)}, \tag{13}$$

with $T_0 = T(X_0) = \overline{T}e^{-\phi X_0}$ the initial –pre-industrial– temperature, (12) becomes

$$\left(\frac{u}{1-\beta}\right)^{-\frac{1-\beta}{\beta}} T_0^{-\frac{m}{\phi}} \frac{1-e^{-m(X_0-X)}}{m} = \frac{1}{\alpha-\beta}\left(K_0^{-\frac{\alpha-\beta}{\beta}} - K^{-\frac{\alpha-\beta}{\beta}}\right) \tag{14}$$

where

$$m = \frac{\varepsilon(1-\beta)+\gamma}{\beta}\phi.$$

The highest sustainable level of utility is obtained by letting $X \to 0$ and $K \to \infty$ in (14). Then we get

$$\left(\frac{u}{1-\beta}\right)^{-\frac{1-\beta}{\beta}} T_0^{-\frac{m}{\phi}} \frac{1-e^{-mX_0}}{m} = \frac{1}{\alpha-\beta} K_0^{-\frac{\alpha-\beta}{\beta}} \tag{15}$$

i.e.

$$u^* = (1-\beta)\left(\frac{(\alpha-\beta)}{m}\right)^{\frac{\beta}{1-\beta}} T_0^{-\frac{\beta m}{(1-\beta)\phi}} \left(1-e^{-mX_0}\right)^{\frac{\beta}{1-\beta}} K_0^{\frac{\alpha-\beta}{1-\beta}}. \tag{16}$$

As in Solow [1974], u^* is all the highest since the initial endowments of the economy in man-made and natural capital are high; moreover, it depends negatively on the initial temperature.

Dividing side by side (14) by (15) yields

$$\left(e^{mX} - 1\right) K^{\frac{\alpha-\beta}{\beta}} = K_0^{\frac{\alpha-\beta}{\beta}} \left(e^{mX_0} - 1\right) = B = \text{constant}. \tag{17}$$

This represents a family of "isoquants" or iso-utility curves for different values of B. We can trace out the level of K required over time as X tends to zero. These "isoquants" turn out to be convex to the origin and asymptotic to the axes in the (K, X) plane, indicating that the model is well-behaved. The economy will follow the iso-utility curve corresponding to the highest sustainable utility level u^* in a downward direction, as man-made capital substitutes for oil.

It follows that

$$\dot{K} = \beta \frac{C(u^*, T(X))}{1-\beta} = \beta \frac{u^* T(X)^{\varepsilon}}{1-\beta} = \beta \frac{u^* T_0^{\varepsilon} e^{\varepsilon \phi (X_0 - X)}}{1-\beta} \tag{18}$$

which tends to a constant as $X \to 0$. Hence we observe the classic result for maximin paths with exhaustible resources, namely that $K(t)$ is linear in time, here only in the limit. Substitution for $K(t)$ above yields

$$\dot{K} = \beta \frac{u^* T_0^{\varepsilon} e^{\varepsilon \phi X_0}}{1-\beta} (B K^{-\frac{\alpha-\beta}{\beta}} + 1)^{-\frac{\varepsilon \beta}{\varepsilon(1-\beta)+\gamma}} \tag{19}$$

which is not possible to integrate with the usual functions.

The case that Stollery solved corresponds to $\varepsilon = 0$ and this case does indeed have $K(t)$ linear in time and c unchanging. We reported on this above.

2.3 Effects of a greater weight on temperature in the utility function

We now investigate more thorougly how the optimal paths are affected by the value of ε, the parameter describing the weight of temperature in the utility function. Temperature in the utility function captures the direct impact of global warming on welfare.

Let us first examine the slope of the iso-utility curves defined by equation (17). This slope is

$$\frac{dX}{dK} = -\frac{\alpha - \beta}{\beta} \frac{1 - e^{-mX}}{m} K \equiv -\frac{\alpha - \beta}{\beta} g(m, X) K.$$

It is easy to check that derivative $g_m(m, X) = \left[(1 + mX)e^{-mX} - 1\right] / m^2$ is negative. The numerator is equal to 0 when $X = 0$ and its derivative with respect to X is $-m^2 X e^{-mX} < 0$. The numerator is therefore negative for any positive X. Thus $g_m(m, X)$ is negative. At any given point in the (K, X) plane, the larger ε, the less steep is the iso-utility curve. As this curve is the trajectory followed by the economy, this confirms that the economy reacts by saving on the resource, on X.

We need however to be careful when making inferences at this point. A higher level of ε implies that the oil stock will be higher for any given level of *capital,* $K(t)$ reached by the economy. It does not tell us that the stock will be higher at any particular *date* under consideration. We must examine the behaviour of $\dot{X}(t)$ and $\dot{K}(t)$.

Both $\dot{K}$ and c are proportional to Y, and $\dot{X}/\dot{K} = dX/dK$, which obviously decreases (in absolute value) through time, as K increases and X decreases.

We show now that an increase in ε implies an initial decrease of production, Y. Using equation (16), initial optimal consumption is

$$c(0) = u^* {T_0}^{\varepsilon} = \Psi g(m, X_0)^{\frac{\beta}{1-\beta}} \tag{20}$$

where

$$\Psi = (1-\beta)\,(\alpha-\beta)^{\frac{\beta}{1-\beta}}\, {T_0}^{-\frac{\gamma}{1-\beta}} K_0^{\frac{\alpha-\beta}{1-\beta}}$$

does not depend on ε. As $g(m, X_0)$ decreases in m, which itself increases with ε, we find that a higher ε induces the economy to have lower levels of consumption, capital accumulation and production, initially. Since $Y(0) = {K_0}^{\alpha} x(0)^{\beta} T(X_0)^{-\gamma}$ where $x(0)$ is the only variable factor, we infer that $x(0)$ decreases[7]. This is rather natural. An economy more concerned with global warming (one with a larger ε) chooses a slower depletion of the resource, and this implies lower production, consumption and capital accumulation "early" in the program.

We now turn to the long run level of production. The long run consumption level, reached when the oil stock is asymptotically exhausted ($X \to 0$) is

$$c_{\infty} = u^* \overline{T}^{\varepsilon} = u^* {T_0}^{\varepsilon} e^{\varepsilon \phi X_0} = c(0) e^{\varepsilon \phi X_0}. \tag{21}$$

It increases with ε if the following inequality holds: $\frac{\beta}{1-\beta} \frac{g_m(m,X_0)}{g(m,X_0)} \frac{\partial m}{\partial \varepsilon} + \phi X_0 > 0$. We see easily that this is equivalent to $mX_0 > 1 - e^{-mX_0}$, which is always true.

It follows from continuity then that an economy with a higher ε exhibits paths of consumption, investment and production lower early in the program and higher later in the program. It means that a society with a maximin social welfare function experiencing a change in preferences (larger ε) optimally sacrifices some goods consumption early on, but will benefit from more consumption later on and forever. The decrease in consumption at the beginning of the trajectory turns on an early decrease in oil extraction and burning, necessary to slow down the temperature increase, early. The temperature rise is more gradual with this larger ε.

3 Zero-discounting paths

We now assume zero discounting and follow the approach introduced by Ramsey [1928]. The objective function becomes

$$\max \int_0^\infty [U(c(t), T(X(t))) - U^*] \, dt,$$

where U^* is the constant long run level of utility, and

$$U(c, T(X)) = \frac{u(c, T(X))^{1-1/\sigma}}{1 - 1/\sigma}, \qquad 0 < \sigma < 1.$$

The instantaneous utility level U depends on a composite consumption index $u(c, T)$ which characterizes intratemporal preferences for consumption and temperature. Parameter σ is the intertemporal elasticity of substitution. We assume σ to be lower than unity. Then $U(c, T)$ is negative and tends to zero if $u(c, T)$ tends to plus infinity. This implies that U^* is equal to zero. We assume ex ante that there exist feasible paths such that $u(c, T)$ tends to infinity as t tends to infinity. We derive optimality conditions, characterize the potential optimal solution and check ex post that this is indeed the case for our specification of the production and utility functions.

The maximin case is obtained as a limit case for $\sigma = 0$ (d'Autume and Schubert [2008a]). Society then refuses any intertemporal substitution and, in regular cases, utility remains constant over time.

We follow here the method proposed by d'Autume and Schubert [2009].

Let $V(K,X)$ be the value function of the problem. Bellman equation is:

$$0 = \max_{c,x} \left[U(c, T(X)) - U^*\right] + V_K(K,X)\left[F(K,x,T(X)) - c\right] - V_X(K,X)x.$$

First order optimality conditions are:

$$U_c = V_K, \qquad V_K F_x = V_X.$$

Substituting in the Bellman equation, we obtain

$$0 = U(c, T(X)) - U^* + U_c(c,T(X))\left[F(K,x,T(X)) - c - xF_x(K,x,T(X))\right].$$

As $F(K,x,T(X)) - c - xF_x(K,x,T(X)) = \dot{K} - qx$, the optimal path satisfies what d'Autume and Schubert [2008b] called a "Keynes-Ramsey-Hartwick" rule:

$$G = \dot{K} + q\dot{X} = \frac{U^* - U(c,T(X))}{U_c(c,T(X))}, \tag{22}$$

with $U^* = 0$. This rule appears as a generalization of both the "invest resource rents" and the Keynes-Ramsey rules. If society has a zero discount rate but is ready to accept intertemporal substitution, genuine savings should not be zero but positive. According to the Keynes-Ramsey rule, its level depends on the distance to the stationary point. More precisely, its value expressed in terms of utility $U_c\left[\dot{K} + q\dot{X}\right]$ is equal to the distance $U^* - U(c,T(X))$ between current utility and its long run value. Thus, the farther the economy from the stationary point, the higher its genuine savings expressed in terms of utility.

As $U = u^{1-1/\sigma}$, we have, with our specification (11) of u,

$$\frac{U_c}{U(c,T(X))} = \left(1 - \frac{1}{\sigma}\right)\frac{u_c}{u(c,T(X))} = \left(1 - \frac{1}{\sigma}\right)\frac{1}{c}.$$

Taking into account $U^* = 0$, equation (22) becomes

$$G = \dot{K} + q\dot{X} = \frac{\sigma}{1-\sigma}c.$$

With our functional forms, this equation reads $\dot{K} = qx + \frac{\sigma}{1-\sigma}c = \beta Y + \frac{\sigma}{1-\sigma}c$ which, together with $\dot{K} = Y - c$, yields the rule of production sharing between consumption and investment along the optimal path, and the value of genuine savings:

$$\begin{aligned} c &= (1-\widehat{\beta})Y, \\ \dot{K} &= \widehat{\beta}Y, \\ G &= \left(\widehat{\beta} - \beta\right)Y, \end{aligned}$$

with

$$1 - \widehat{\beta} = (1-\beta)(1-\sigma).$$

The saving rate $\widehat{\beta}$ lies between β and 1, and is equal to β when $\sigma = 0$. Consumption, investment and genuine savings all are constant shares of the gross national product. In the maximin case, $\sigma = 0$ and $\widehat{\beta} = \beta$ so that genuine savings is equal to zero. When $\sigma > 0$, genuine savings is positive and even growing, as we shall check that Y is increasing. This of course does not contradict the Keynes-Ramsey-Hartwick rule as marginal utility U_c decreases so that genuine savings expressed in terms of utility decreases to zero.

We now proceed with the other optimality conditions.

The envelop theorem allows us to obtain the evolution of the shadow prices. Let

$$\lambda = U_c(c, T(X)) = V_K, \quad \mu = V_X.$$

The price of the resource stock in terms of capital is $q = \mu/\lambda$. Differentiating the Bellman equation with respect to K, we get

$$0 = V_{KK}\left[F(K, x, T(X)) - c\right] + V_K F_K(K, x, T(X)) - V_{KX}\, x$$

i.e.

$$\dot{\lambda} + F_K \lambda = 0, \tag{23}$$

and, differentiating with respect to X,

$$0 = U_T(c, T(X))T'(X) + V_{KX}\left[F(K, x, T(X)) - c\right] + V_K F_T(K, x, T(X))T'(X) - V_{XX}\, x$$

i.e.

$$\left(U_T + U_c F_T\right) T'(X) + \dot{\mu} = 0. \tag{24}$$

We deduce the evolution of q i.e. the Hotelling rule:

$$\frac{\dot{q}}{q} = F_K - \frac{1}{q}\frac{\left(U_T + U_c F_T\right) T'(X)}{U_c}. \tag{25}$$

From the definition of U, the marginal rate of substitution U_T/U_c only depends on the $u(c, T(X))$ function and is equal to u_T/u_c. Equation (25) is then exactly equation (1) of the maximin case.

With Cobb-Douglas production and utility functions, equations (23) and (24) take the form

$$\frac{\dot{\lambda}}{\lambda} = -\alpha\frac{Y}{K} = -\frac{\alpha}{\widehat{\beta}}\frac{\dot{K}}{K},$$

$$\frac{\dot{\mu}}{\mu} = -\frac{(U_T + U_c F_T)T'(X)}{U_c F_x} = \frac{(\varepsilon c + \gamma Y)}{\beta\frac{Y}{x}}\frac{T'(X)}{T(X)} = -\widehat{n}\frac{\dot{T}}{T},$$

where

$$\widehat{n} = \frac{\varepsilon(1 - \widehat{\beta}) + \gamma}{\beta}.$$

Thus the shadow price of capital can be expressed as a function solely of the capital stock, while the shadow price of the resource can be expressed as a function solely of the temperature:

$$\lambda = B_1 K^{-\frac{\alpha}{\beta}}, \qquad \mu = B_2 T(X)^{-\widehat{n}},$$

where B_1 and B_2 are constants.

The values of the capital and the resource stocks are

$$
\begin{aligned}
W_K &= \lambda K = B_1 K^{1-\frac{\alpha}{\widehat{\beta}}}, \\
W_X &= \mu X = B_2 X T(X)^{-\widehat{n}}.
\end{aligned}
$$

W_K tends to zero as time tends to infinity iff $\alpha < \widehat{\beta}$ and K tends to zero, or $\alpha > \widehat{\beta}$ and K tends to infinity. If the capital stock were to tend to zero, production and consumption would do the same as the resource input also has to tend to zero. This cannot be optimal in a model with zero discounting. Then an optimal path exists if and only if

$$
\alpha > \widehat{\beta} \Longleftrightarrow \alpha > \sigma + \beta(1-\sigma).
$$

This condition is more stringent that the condition $\alpha > \beta$ required in the maximin case. Moreover, it involves technological parameters only in the maximin case, whereras it involves here both technological and preference parameters. Along the optimal path, the capital stock grows without limit, in order to maintain an increasing consumption in spite of the decrease in resource use.

W_X tends to zero as time tends to infinity if and only if $XT(X)^{-\widehat{n}}$ tends to zero. As $T(X)$ must be finite, X has to tend to zero. The resource is asymptotically exhausted along the optimal path.

The Hotelling rule now reads

$$
\frac{\dot{q}}{q} = \frac{\alpha}{\widehat{\beta}} \frac{\dot{K}}{K} - \widehat{n} \frac{\dot{T}}{T}.
$$

This shows that q can be expressed as a simple function of K and $T(X)$:

$$
q = \Phi_0 K^{\frac{\alpha}{\widehat{\beta}}} T(X)^{-\widehat{n}}, \tag{26}
$$

with

$$
\Phi_0 = q_0 K_0^{-\frac{\alpha}{\widehat{\beta}}} T(X_0)^{\widehat{n}}. \tag{27}
$$

As in the maximin case, the solution is obtained by time elimination and variable separation.

The ratio of the two equations of motion $\dot{K} = \widehat{\beta} Y$ and $\dot{X} = -x = -\beta \frac{Y}{q}$ yields, using (26):

$$\frac{dX}{dK} = -\frac{\beta}{\widehat{\beta}} \frac{1}{q} = -\frac{1}{\Phi_0} \frac{\beta}{\widehat{\beta}} K^{-\frac{\alpha}{\widehat{\beta}}} T(X)^{\widehat{n}}.$$

Thus we obtain

$$T(X)^{-\hat{n}} dX = -\frac{1}{\Phi_0} \frac{\beta}{\widehat{\beta}} K^{-\frac{\alpha}{\widehat{\beta}}} dK,$$

which integrates in

$$\int_{X_0}^{X} T(\xi)^{-\hat{n}} d\xi = \frac{1}{\Phi_0} \frac{\beta}{\alpha - \widehat{\beta}} \left(K^{1-\frac{\alpha}{\widehat{\beta}}} - K_0^{1-\frac{\alpha}{\widehat{\beta}}} \right).$$

In order to get an explicit solution we revert to Stollery's temperature function (6) so that the previous equation becomes

$$\frac{\bar{T}^{-\hat{n}}}{\widehat{m}} \left(e^{\widehat{m} X} - e^{\widehat{m} X_0} \right) = \frac{1}{\Phi_0} \frac{\beta}{\alpha - \widehat{\beta}} \left(K^{1-\frac{\alpha}{\widehat{\beta}}} - K_0^{1-\frac{\alpha}{\widehat{\beta}}} \right), \tag{28}$$

with

$$\widehat{m} = \widehat{n} \phi.$$

Making $X \to 0$ and $K \to \infty$ in equation (28) then yields, as $\alpha > \widehat{\beta}$,

$$\frac{\bar{T}^{-\hat{n}}}{\widehat{m}} \left(1 - e^{\widehat{m} X_0} \right) = -\frac{1}{\Phi_0} \frac{\beta}{\alpha - \widehat{\beta}} K_0^{1-\frac{\alpha}{\widehat{\beta}}}, \tag{29}$$

which determines Φ_0 and therefore q_0, by equation (27):

$$q_0 = \Phi_0 K_0^{\frac{\alpha}{\widehat{\beta}}} e^{\widehat{m} X_0} = \bar{T}^{\hat{n}} \frac{\beta \widehat{m}}{\alpha - \widehat{\beta}} \frac{K_0}{1 - e^{-\widehat{m} X_0}}. \tag{30}$$

We also have, from (28),

$$\frac{\bar{T}^{-\hat{n}}}{\widehat{m}} \left(e^{\widehat{m} X} - 1 \right) = \frac{1}{\Phi_0} \frac{\beta}{\alpha - \widehat{\beta}} K^{1-\frac{\alpha}{\widehat{\beta}}}. \tag{31}$$

Dividing side by side equations (31) and (29) allows us to obtain

$$K^{\frac{\alpha-\widehat{\beta}}{\widehat{\beta}}}\left(e^{\widehat{m}X}-1\right)=K_0^{\frac{\alpha-\widehat{\beta}}{\widehat{\beta}}}\left(e^{\widehat{m}X_0}-1\right), \tag{32}$$

which shows that an aggregate of the capital and natural resource stocks is conserved along the optimal path:

Equation (32) defines a family of trajectories in the (K, X) plane. Initial endowments (K_0, X_0) determine the relevant trajectory. As in the maximin case, the economy follows this curve in a downward direction, as man-made capital substitutes for natural capital. The capital stock tends to infinity, as the resource stock tends to zero. In the present zero-discounting case, however, these trajectories are no more iso-utility curves. To the contrary, as we check below, utility is growing along each one of these trajectories whenever $\sigma \neq 0$.

We now characterize the evolution of K.

From (26), (29) and (32) we obtain

$$\begin{aligned}\frac{q}{K} &= \Phi_0 K^{\frac{\alpha}{\beta}-1}e^{\widehat{m}X}=\bar{T}^{\hat{n}}\frac{\beta\widehat{m}}{\alpha-\widehat{\beta}}\frac{K_0^{1-\frac{\alpha}{\beta}}}{e^{\widehat{m}X_0}-1}K^{\frac{\alpha}{\beta}-1}e^{\widehat{m}X}\\ &= \bar{T}^{\hat{n}}\frac{\beta\widehat{m}}{\alpha-\widehat{\beta}}\frac{1}{e^{\widehat{m}X_0}-1}\left(\frac{K_0}{K}\right)^{1-\frac{\alpha}{\beta}}\left[1+\left(\frac{K_0}{K}\right)^{\frac{\alpha}{\beta}-1}\left(e^{\widehat{m}X_0}-1\right)\right].\end{aligned}$$

On the other hand, from the production function (5) and $qx = \beta Y$, we have

$$q=\beta Y^{\frac{1-\beta}{\beta}}K^{-\frac{\alpha}{\beta}}\bar{T}^{-\frac{\gamma}{\beta}}e^{\frac{\gamma\phi}{\beta}X}.$$

Using (32) again yields

$$\bar{T}^{\hat{n}}\frac{\widehat{m}}{\alpha-\widehat{\beta}}\frac{1}{e^{\widehat{m}X_0}-1}\left(\frac{K_0}{K}\right)^{1-\frac{\alpha}{\beta}}\left[1+\left(\frac{K_0}{K}\right)^{\frac{\alpha}{\beta}-1}\left(e^{\widehat{m}X_0}-1\right)\right]^{1-\frac{\gamma\phi}{\beta\widehat{m}}}=Y^{1-\frac{1}{\beta}}K^{\frac{\alpha}{\beta}-1}\bar{T}^{-\frac{\gamma}{\beta}}.$$

Then

$$Y=\widehat{A}\left[1+\widehat{B}K^{1-\frac{\alpha}{\beta}}\right]^{-\frac{\varepsilon(1-\widehat{\beta})}{\varepsilon(1-\widehat{\beta})+\gamma}\frac{\beta}{1-\beta}}K^{\frac{\alpha(\widehat{\beta}-\beta)}{\widehat{\beta}(1-\beta)}},$$

where $\widehat{A} = \left(\frac{(\alpha-\widehat{\beta})\widehat{B}}{\widehat{m}\overline{T}^{\widehat{n}}}\right)^{\frac{\beta}{1-\beta}} \overline{T}^{-\frac{\gamma}{1-\beta}}$ and $\widehat{B} = K_0^{\frac{\alpha}{\widehat{\beta}}-1}\left(e^{\widehat{m}X_0} - 1\right)$ are positive constants. This equation yields, using $\dot{K} = \widehat{\beta}Y$, a differential equation in K, impossible to integrate with the usual functions. It is easy to check that this equations reduces to (19) when $\sigma = 0$.

We now turn to the expression of u. We have

$$\begin{aligned} u &= cT(X)^{\varepsilon} = (1-\widehat{\beta})Y\overline{T}^{\varepsilon}e^{-\varepsilon\phi X} \\ &= (1-\widehat{\beta})\overline{T}^{\varepsilon}\widehat{A}\left[1+\widehat{B}K^{1-\frac{\alpha}{\widehat{\beta}}}\right]^{-\frac{\varepsilon\beta}{\varepsilon(1-\widehat{\beta})+\gamma}\left(1+\frac{1-\widehat{\beta}}{1-\beta}\right)} K^{\frac{\alpha(\widehat{\beta}-\beta)}{\widehat{\beta}(1-\beta)}}. \end{aligned}$$

As $\alpha > \widehat{\beta} > \beta$, u is an increasing function of K and therefore increases without limit along the optimal path. Utility $U = u^{1-1/\sigma}/(1-1/\sigma)$ increases and tends to zero as K and time tend to infinity. We have indeed checked ex post that $U^* = 0$.

By definition, the elasticity of substitution controls the amount of intertemporal substitution. Thus an economy with a high σ accepts to sacrifice current utility in order to increase future utility. To the contrary, an economy with a zero σ chooses a constant utility level.

4 Concluding Remarks

Thus Stollery's global warming model represents an interesting and elegant extension of Solow [1974]. Increased consumption compensates agents for the disutility implied by the unavoidable rise in temperature and allows them to maintain a constant level of utility. We have been able to complement Stollery's analysis and to provide a closed-form solution for the case where temperature also has a direct effect on utility. This allowed us to identify the role played by a temperature externality in utility. It leads to less physical production and consumption in the short run, in order to save the resource and prevent an excessive rise in temperature. But it also boosts consumption and production later on and forever, as capital accumulation benefits from the postponement of the rise in temperature.

We also considered a more general zero-discounting criterion which enables us to examine the effects of greater acceptance of intergenerational substitution of welfare levels, and show that it plays to the advantage of future generations. To the contrary, a maximin criterion appears to protect current generations.

Our results are dependent on the two assumptions of Cobb-Douglas production and utility. The first one captures in a simple way the essential role of the resource as it states that it is impossible to produce without resource. This does not prevent the resource stock from converging to zero, but only at the limit when time goes to infinity and an infinite amount of produced capital, in the limit of time, become available to substitute for the vanishing resource. The second one is more restrictive and an interesting extension would be to consider other specification ot the utility function, such as a CES utility function, with limited substitution between consumption and the temperature disamenity. As shown in a slightly different context in d'Autume and Schubert [2008a], the optimal solution would then be to maintain forever some finite portion of the oil stock, an amount endogenously determined, thus preventing the temperature from reaching its maximal level. Other specifications of the temperature function should also be consdered.

Notes

[1] We honor the memory of researcher Ken Stollery (1948-2005) who was struck down by illness at a tragically early age.

[2] Hamilton and Ulph [1995] developed a somewhat different version of a Solow model with global warming independently of Stollery.

[3] This assumption neglects the phenomenon of absorption of carbon dioxide by natural sinks, which is possibly an important feature of global warming (see for instance Farzin and Tahvonen [1996]), but is very slow, uncertain and probably decreasing with the increase of carbon concentration. It amounts to consider that carbon accumulation is irreversible.

[4] This version of Hotelling's Rule could be derived in a variety of optimal growth frameworks. Stollery dervied it for a constant utility objective function via a route developed by Leonard and Long [1992; pp. 300-304].

[5] D'Autume and Schubert [2008a] developed a new approach for generating optimal paths which are maximin. They set out a discounted utilitarian problem with a parameter characterizing the substitutability of utility levels across "adjacent periods" (in continuous time) and proceeded to consider the limiting solution as the substitution parameter tended to zero. The limiting solution is a maximin path. We have omitted this step but we still speak here of a maximin path which is optimal.

[6]D'Autume and Schubert [2008a] show that a CES utility function implies that society will choose to conserve forever a strictly positive level of resource, which is endogenously determined.

[7]This result also derives from the fact that $x(0)/\dot{K}(0) = -dX(0/dK(0)$ decreases with ε.

References

[1] D'Autume, Antoine and Katheline Schubert [2008a] "Hartwick's Rule and Maximin Paths when the Exhaustible Resource has an Amenity Value", *Journal of Environmental Economics and Management*, 56, 3, pp. 260–274.

[2] D'Autume, Antoine and Katheline Schubert [2008b] "Zero Discounting and Optimal Paths of Depletion of an Exhaustible Resource with an Amenity Value", *Revue d'Economie Politique*, forthcoming.

[3] Dasgupta, Partha, and Geoffrey Heal [1979] *Economic theory and exhaustible resources*, Cambridge University Press.

[4] Dixit, Avinash K., Peter Hammond, and Michael Hoel [1980] "On Hartwick's Rule for Regular Maximin Paths of Capital Accumulation and Resource Depletion", *Review of Economic Studies*, 47, 3, pp. 551–56.

[5] Farzin, Y. Hossein and Olli Tahvonen [1996] "Global Carbon Cycle and the Optimal Time Path of a Carbon Tax", *Oxford Economic Papers*, 48, 4, pp. 515–536.

[6] Hamilton, Kirk and David Ulph [1995] "The Hartwick Rule in a Greenhouse World", unpublished manuscript, University College, London.

[7] Hartwick, John [1977] "Intergenerational Equity and the Investing of Rents from Exhaustible Resources", *American Economic Review*, 66, 5, pp. 972–974.

[8] Leonard, Daniel and Ngo Van Long [1992] *Optimal Control Theory and Static Optimization in Economics*, New York: Cambridge University Press.

[9] Mitra, Tapan [1980] "On Optimal Depletion of Exhaustible Resources: Existence and Characterization Results", *Econometrica*, 48, 6, pp. 1431–1450.

[10] Ramsey, Frank [1928] "A Mathematical Theory of Savings", *Economic Journal*, 38, pp. 543–559.

[11] Solow, Robert M. [1974] "Intergenerational Equity and Exhaustible Resources", *Review of Economics Studies*, Symposium volume, pp. 29–46.

[12] Stollery, Kenneth R. [1998] "Constant Utility Paths and Irreversible Global Warming", *Canadian Journal of Economics*, 31, 3, pp. 730–42.